OPTIMIZATION

The Bellman Equation

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Optimization

Optimization is **static** or **dynamic**, and *unconstrained* or *constrained*.
Further, optimization is concerned with search for local or global optima.

Static optimization is typically single stage and time independent. Either models or at least data for possible models are available. If models exists, either derivates exist or they don't. If derivates exist, unconstrained optimization is typically first (e.g. gradient descent) or second order, and suitable for local optima. Under constrains, usually different search strategies must be adopted. If models don't exist, optimization is usually curve or multi-dimensional model fitting, and quality of models depends on quality of data.

Dynamic optimization is used for multistage and time-varying action and decision-making, as typically appearing in optimization problems related to consumption, investment, portfolio and pricing.

Investments

The total wealth (or asset, or budget) at time t is $W(t)$.

Invest part of your wealth on two different targets:

$$V_1(t) \quad V_2(t)$$

$$V_{total}(t) = V_1(t) + V_2(t) \leq W(t)$$

$V_1(t)$ may be a function of $V_2(t)$:

$$V_2(t) = F(V_1(t))$$

e.g. like

$$V_2(t) = V_{total}(t) - V_1(t)$$

in the case $V_{total}(t)$ has been fixed.

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Bellman's Principle of Optimality

An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

In accordance with this principle, subsequent investments may be arranged as functions of previous investments:

$$V_j(t+1) = G_j(V_j(t))$$

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Return of Investment

The expected return of your investment: $R_j(V_j(t))$

Your expected return can decrease over time, using a discount factor: $0 < \delta < 1$

Then

$$\tilde{R}_j(t_n) = \delta^n R_j(V_j(t_n))$$

or

$$\tilde{R}_j(t_n) = e^{-f(\delta,n)} R_j(V_j(t_n))$$

where $f(\delta,n)$ increases with n and $f(\delta,0) < 1$.

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Your total return can be defined as

$$R_{total}(t_n) = \tilde{R}_1(t_n) + \tilde{R}_2(t_n)$$

or

$$R_{total}(t_n) = \alpha\tilde{R}_1(t_n) + \beta\tilde{R}_2(t_n)$$

with $\alpha + \beta = 1$.

Optimal Investment Strategy

Maximizing your values then means optimizing with respect to your starting point:

$$
\begin{aligned}
M(t_0) &= \max_{F,G} \sum_{i=0}^{\infty} R_{total}(t_i) \\
&= \max_{F,G} R_{total}(t_0) + \delta M(t_1)
\end{aligned}
$$

This is the **Bellman Equation** (for this particular problem) in its functional and recursive form.

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The Bellman Equation Computationally

Sometimes M can be solved analytically, but mostly M must be computed.

Then find t_0 so that $M(t_0)$ reaches its maximum, and you're done!

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Dynamic Optimization Project Phases

Phases and steps:

- ✓ INVENTORY
  - ✓ Understand the nature of your optimization problem.
  - ✓ Organize your data and check availability of possible submodels.
- ✓ SPECIFICATION
  - ✓ Identify variables, states and parameters.
  - ✓ Evaluate alternatives for the objective function.
- ✓ CREATION
  - ✓ Formulate your optimization problem.
  - ✓ Find your Bellman Equation.
- ✓ OPTIMIZATION

Four Computing Oy
Mörlingintie 10
FIN-24910 Halikko
Finland

patrik.eklund@fourcomp.com
http://www.fourcomp.com/oak/
+358 50 343 1403

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